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Damage Identification of Structures Including System Uncertainties and Measurement Noise

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This paper proposes a statistical method for the damage identification of structures based on the measured acceleration response with reference to an analytical model. Uncertainties in the system parameters, such as the structural parameters of the finite element model, the excitation force acting on the structure, and the measured acceleration response from the perturbed state of the structure, are discussed and they are included in the study of the damage identification. Each of these uncertainties is assumed to have a zero mean and is normally distributed. The propagation of each of these uncertainties in an updating damage-identification algorithm is then studied, based on the response-sensitivity approach. A three-dimensional, five-bay, steel-frame structure with local damage in two members is studied for illustration. The mean and standard deviations of the local stiffness parameters identified compare favorably with those from the Monte Carlo technique. The probability density function of the identified local stiffness parameters in both the intact and perturbed states is compared in a subsequent reliability assessment.

Nomenclature

C	=	damping matrix
$\mathrm{cov}(X_1, X_2)$	=	covariance of the random variables X_1 and X_2
D	=	mapping vector
E_i	=	true elastic modulus of material of the ith element
$\tilde{E}_i$	=	measured elastic modulus of material of the ith element
Ep	=	magnitude of uncertainty
$E(\cdot)$	=	expectation value
f_i	=	true value of the excitation at the ith time instance
$\tilde{f}_i$	=	measured value of the excitation at the ith time instance
$F(t)$	=	force excitation vector
K	=	stiffness matrix
K_i	=	ith elemental stiffness matrix
M	=	mass matrix
N	=	a vector with standard normal random distribution
S	=	sensitivity matrix
$\mathrm{var}(\cdot)$	=	variance value
X_{E_i}	=	random variable associated with the elastic modulus of material of the ith element
X_{F_i}	=	random variable associated with the measured excitation force at the ith time instance
X_p	=	random variable associated with the structural parameters
X_{ρ_i}	=	random variable associated with the material density of the ith element
$X_{\ddot{x}_{di}}$	=	random variable associated with the measured acceleration from the damaged structure at the ith time instance
x	=	displacement vector
$\dot{x}$	=	velocity vector
$\ddot{x}$	=	acceleration vector
$\ddot{x}_d$	=	acceleration responses measured directly from the damaged structure
$\ddot{x}_l^k$	=	updated acceleration responses at l degrees of freedom after the kth iteration
$\ddot{x}_u$	=	analytical acceleration responses from the intact structure
α_i	=	fractional parameter of the ith element
α^k	=	fractional parameter vector after the kth iteration
α^0	=	fractional parameter vector with the initial analytical model
$\Delta\alpha$	=	damage vector or vector of small changes in the structural parameter
$\Delta\alpha^k$	=	fractional change in the parameter vector after the kth iteration
μ_x	=	mean of variable x
ρ_i	=	true material density of the ith element
$\tilde{\rho}_i$	=	measured material density of the ith element
σ_x	=	standard deviation of variable x
$\Phi(\cdot)$	=	normal distribution.
$\phi(x)$	=	probability density function of variable x

I. Introduction

EARLY detection of damage in engineering systems during their service life has been receiving increasing attention from researchers in the last few decades. Information on the location and extent of damage could assist in the diagnosis of the structural health conditions and in the recommendation on associated maintenance work. The occurrence of damage in a structure produces changes in its global static and dynamic characteristics. Natural frequencies and mode shapes are usually taken as the measured information to identify local damages in frequency domain. However, Farrar et al. [1] showed that the shift of natural frequencies was not sufficiently sensitive to detect local damage in a highway bridge. Another study [2] showed that there are still factors that prevent the general application of the mode-shape-based damage-identification methods.

There is also literature on damage identification directly using structural dynamic response without the need of a modal-extraction procedure. Cattarius and Inman [3] used the phase shift in the time history of the structural vibration response to identify the presence of damage in smart structures. Choi and Stubbs [4] formed the damage index directly from the time response to locate and quantify damage in a structure. Lu and Gao [5] proposed a new method for damage diagnosis using time-series analysis of vibration signals, which is formulated in a novel form of autoregressive with exogenous input

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(ARX) model with acceleration-response signals. Kang et al. [6] presented a system identification scheme in the time domain to estimate the stiffness and damping parameters of a structure using measured accelerations, and the method is demonstrated with numerical study on a two-span truss bridge and a laboratory study on a three-story shear building model.

More recently Law et al. [7] developed the sensitivity-based damage-identification method basedon the wavelet packet energy of the measured accelerations, and the method can identify local damage of a structure from a few measurement locations. Later, the sensitivity matrix of response with respect to a system parameter was derived analytically for damage identification [8,9]. But the damage-identification results are subject to the effect of measurement noise and model error. These effects on damage identification are further studied with the sensitivity of the wavelet coefficient from the structural response [10], with much better results. To reduce the effect of uncertainty in the excitation, unit impulse response is directly considered (instead of the time response) in the damage-identification process [11].

In practice, noise exists in the measured vibration data and there are also uncertainties with the initial analytical model, leading to incorrect damage identification. Many current research efforts are on the study of the influence of these uncertainties on the results of vibration-based damage identification [12–17]. Some of them have considered the effect of uncertainty on the model updating [13,14], in which measured statistical changes in the natural frequencies and mode shapes, along with a correlated analytical stochastic finite element model, are used to assess the integrity of a structure. Later, Xia et al. [15,16] developed a statistical damage-identification algorithm based on the change of natural frequency and mode shape to account for the effects of random noise in the vibration data and variations in the finite element model. The statistics of the parameters are estimated by the perturbation method and verified with the Monte Carlo technique. However, all existing works are on the uncertainty with the system parameter and modal parameters. The effect of uncertainties from all practical influencing parameters of the system in the time-domain damage identification has not been investigated.

A statistical method for structural damage identification based on a measured acceleration response with a reference model is proposed in this paper. Uncertainties in the system parameters, such as the structural parameters of the finite element model, the excitation force acting on the structure and the measured acceleration response from the perturbed state of the structure, are analyzed. An analytical formula is given and included in the study for the damage identification. Each of these uncertainties is assumed to have a zero mean and is normally distributed. The effect of each of them on the assessment results is studied in the process of an updating damage identification based on the response-sensitivity approach. A three-dimensional, five-bay, steel-frame structure with local damage in two members is studied for illustration. The mean value and standard deviation of the identified local stiffness parameters compare favorably with those from the Monte Carlo technique. Results obtained show that the proposed method is valid, though computationally demanding. The probability density functions (PDFs) of the identified local stiffness parameters in both the intact and perturbed states are compared in a subsequent reliability assessment.

II. Theoretical Formulation

A. Damage-Identification Procedure

The procedure of damage identification can be found in detail in [9] and is briefly described next. The equation of motion of an N-degree-of-freedom (DOF) damped structural system under general excitation is given as

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = D \cdot f(t) \tag{1}$$

where M, C, and K are the $N \times N$ mass, damping, and stiffness matrices, respectively; D is the constant-mapping vector linking the force input to the corresponding DOF of the system; $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ are the displacement, velocity, and acceleration vectors, respectively; and $f(t)$is the input force excitation vector. The displacement and its derivatives can be computed from Eq. (1) using the Newmark method (see the Appendix).

We consider a general structure that behaves linearly before and after the occurrence of damage, for the illustration of the proposed approach. The study of a nonlinear system with a time-varying damage model is not included in this study, because it involves a different treatment of the identification problem.

The stiffness matrix of the intact structure, K, is expressed as the summation of elemental stiffnesses K_i of each of the elements as

$$K = \sum_{i=1}^{\mathrm{ne}} K_i$$

where ne is the number of elements in the structure. Because this work focuses on the statistical analysis in the damage-identification technique, an approximate model on the local damage is adopted in which the damaged structural system is expressed as

$$K_d = \sum_{i=1}^{\mathrm{ne}} \alpha_i K_i$$

where α_i is the fractional stiffness of an element, typically with $1.0 \geq \alpha_i \geq 0.0$.

The damage-identification equation based on the acceleration-response sensitivity [9] is give as

$$S \cdot \Delta\alpha = \Delta\ddot{x} = \ddot{x}_d - \ddot{x}_u \tag{2}$$

where

$$S = \begin{bmatrix} \frac{\partial \ddot{x}_l}{\partial \alpha_1} & \frac{\partial \ddot{x}_l}{\partial \alpha_2} & \cdots & \frac{\partial \ddot{x}_l}{\partial \alpha_m} \end{bmatrix} \tag{3}$$

is the sensitivity matrix of the acceleration response $\ddot{x}_l$ at the lth DOF with respect to a change in the structural parameters and is also a function of time; m is the number of the structural parameters to be identified with $m \leq \mathrm{ne}$ in general (but $m = \mathrm{ne}$ is adopted in the present study); $\Delta\alpha$ is the vector of ne fractional change in the stiffness of the system; and $\ddot{x}_d$ and $\ddot{x}_u$ are vectors of the acceleration response at the lth DOF of the damaged and intact structures, respectively. In general, the acceleration response $\ddot{x}_u$ from the physical intact structure is computed from the associated analytical model (reference model) by a dynamic analysis, and $\ddot{x}_d$ is measured directly from the damaged structure.

The physical structure is regularly monitored against the analytical model. If the physical model appears to have different properties when the reference model is evaluated, the analytical model is used and updated iteratively to compute an associated analytical model. The difference between the parameters of the analytical models in the reference and the analytical model at the end of the updating process is the predicted set of damage.

The damage vector $\Delta\alpha$ can be obtained from Eq. (2) using the least-squares method as

$$\Delta\alpha = (S^T S)^{-1} S^T (\ddot{x}_d - \ddot{x}_u) \tag{4}$$

The sensitivity matrix S can be computed by taking the first differential of Eq. (1) with respect to the structural parameter α_i as

$$M\frac{\partial \ddot{x}}{\partial \alpha_i} + C\frac{\partial \dot{x}}{\partial \alpha_i} + K\frac{\partial x}{\partial \alpha_i} = -\frac{\partial K}{\partial \alpha_i}x - \frac{\partial C}{\partial \alpha_i}\dot{x} \tag{5}$$

It is noted that the displacement and its derivatives were obtained from Eqs. (A1) and (A2), and the right-hand-side of Eq. (5) becomes an equivalent forcing term. The displacement and acceleration sensitivities are then obtained from Eq. (5) using the Newmark method. The procedure of the iterative computation to get $\Delta\alpha$ is described as follows:

Step 1) Solve Eq. (1) at the $(k+1)$th iteration with known α^k for $\ddot{x}_u$ and compute $\Delta\ddot{x}$.

Step 2) Solve Eq. (5) at the $(k+1)$th iteration with known $\alpha^k$for $\partial\ddot{x}_l/\partial\alpha$ to get the sensitivity matrix S.

Step 3) Find $\Delta\alpha^{k+1}$ from Eq. (4).

Step 4) Update the stiffness parameter vector by the formula

$$\alpha^{k+1} = \alpha^k + \Delta\alpha^{k+1} \tag{6}$$

Step 5: Repeat steps 1 to 4 until both of the following two criteria are met:

$$\frac{\|(\ddot{x}_l)^{k+1} - (\ddot{x}_l)^k\|}{\|(\ddot{x}_l)^{k+1}\|} \le \text{toler1}, \qquad \frac{\|\alpha^{k+1} - \alpha^k\|}{\|\alpha^{k+1}\|} \le \text{toler2} \tag{7}$$

The tolerances are equal to 1.0×10^{-6} in this paper, unless otherwise stated.

B. Uncertainties of the System

The preceding sensitivity algorithm was developed based on the assumption that both the finite element model and the measured dynamic characteristics are accurate. In practice, errors in the damage-identification procedure always exist, which generally include the discretization error, configuration error, mechanical parameter errors, and measurement errors. These errors may be divided into two categories: biased (systematic) error and random error. The biased error may not have zero mean and may have different types of distributions. Random error, on the other hand, has zero mean and is usually modeled as normally distributed [18].

Because an initial analytical model is used in the preceding sensitivity method, random errors in the structural parameters will be introduced into the stiffness matrix, mass matrix, and even the damping matrix, leading to errors in the damage-identification results. The acceleration response from the reference analytical model of the structure is computed with the same force excitation as in the damaged state, and the force excitation needs to be measured from the damaged structure, with noise included in the measurement. The acceleration response $\ddot{x}_d$ in Eq. (2) is directly measured from the damaged state. The measured acceleration response will contain noise and it will be included in the statistical analysis. Because a deterministic model-updating technique is used in the proposed damage-identification procedure, all the preceding random errors will propagate in the computation system with iterations. They are analyzed further in the remainder of this paper to see how these errors would erode the identification results. A probability prediction can then be obtained for the identified parameter changes for used in the subsequent reliability analysis.

The preceding different random variables are all assumed to have zero mean and normally distributed. They are also assumed to be independent, with no coupling effect in the following studies, and X_p denotes the random variables associated with the structural parameters of the initial analytical model. Broadly speaking, many parameters such as mass density, geometric parameters, and the material elastic modulus, etc., can be regarded as an uncertain parameter of the initial analytical model. However, only the parameters associated with the mass (the material density) and stiffness (the elastic modulus of material) are studied for illustration of the statistical analysis and they are denoted by X_{ρ_i} and X_{E_i}, respectively, for the ith element as

$$\tilde{\rho}_i = \rho_i(1 + X_{\rho_i}), \qquad \tilde{E}_i = E_i(1 + X_{E_i}) \tag{8}$$

where $\tilde{}$ denotes the measured value including the uncertainty, ρ_i and E_i denote the true values, and i denotes the ith element. Every finite element is assigned with these random variables representing the uncertainties in the mass and stiffness properties of the element. The uncertainties with other structural parameters can be similarly defined.

The second type of uncertainty arising from the measured exciting force is denoted with the random variable X_f, and the measured force excitation vector including the uncertainty is related to the random variables as

$$\tilde{f}_i = f_i + X_{f_i} \tag{9}$$

where f_i denotes the true excitation value at the ith time instance.

Similarly, the third type of uncertainty arising from the measured acceleration response is denoted with the random variable $X_{\ddot{x}}$ and it is related to the measured acceleration-response vector, including the uncertainty as

$$\tilde{\ddot{x}}_{di} = \ddot{x}_{di} + X_{\ddot{x}_{di}} \tag{10}$$

where $\ddot{x}_{di}$ denotes the true value of the measured response at the ith time instance from the damaged structure.

C. Derivatives of Local Damage with Respect to the Uncertainties

The following paragraphs present the derivation of relationships between the local damage vector $\Delta\alpha$ identified from Eq. (4) and the different types of uncertainties.

The first derivation of Eq. (2) with respect to the general random variables X is given as

$$\frac{\partial S}{\partial X} \cdot \Delta\alpha + S \cdot \frac{\partial \Delta\alpha}{\partial X} = \frac{\partial \tilde{\ddot{x}}_d}{\partial X} - \frac{\partial \ddot{x}_u}{\partial X} \tag{11}$$

Substituting Eq. (4) into Eq. (11), $\partial\Delta\alpha/\partial X$ can be obtained as

$$\frac{\partial \Delta\alpha}{\partial X} = (S^T S)^{-1} S^T \left(\frac{\partial \tilde{\ddot{x}}_d}{\partial X} - \frac{\partial \ddot{x}_u}{\partial X} - \frac{\partial S}{\partial X} \cdot (S^T S)^{-1} S^T (\tilde{\ddot{x}}_d - \ddot{x}_u) \right) \tag{12}$$

Equation (12) gives the general relationship between an uncertainty and the identified local damage vector. Matrix S and response vector $\ddot{x}_u$ can be obtained from the analytical model. Response vector $\tilde{\ddot{x}}_d$ is obtained from measurement. The unknown terms $\partial\tilde{\ddot{x}}_d/\partial X$, $\partial\ddot{x}_u/\partial X$, and $\partial S/\partial X$ are obtained next for each type of uncertainty for the solution of Eq. (12).

1. *Uncertainties in the System Parameter*

When we consider the uncertainty in the system structural parameter $X = X_p$, we have $\partial\tilde{\ddot{x}}_d/\partial X_p = 0$, because the measured response is independent of the system parameter. Equation (12) can be rewritten as

$$\frac{\partial \Delta\alpha}{\partial X_p} = -(S^T S)^{-1} S^T \left(\frac{\partial \ddot{x}_u}{\partial X_p} + \frac{\partial S}{\partial X_p} \cdot (S^T S)^{-1} S^T (\tilde{\ddot{x}}_d - \ddot{x}_u) \right) \tag{13a}$$

The sensitivity of the analytical response with respect to X_p, $\partial\ddot{x}_u/\partial X_p$, can be obtained by taking the first derivation of Eq. (1) with respect to X_p as

$$M\frac{\partial \ddot{x}}{\partial X_p} + C\frac{\partial \dot{x}}{\partial X_p} + K\frac{\partial x}{\partial X_p} = -\frac{\partial M}{\partial X_p}\ddot{x} - \frac{\partial C}{\partial X_p}\dot{x} - \frac{\partial K}{\partial X_p}x \tag{14}$$

The sensitivity $\partial\ddot{x}_u/\partial X_p$ can be computed [9] from Eqs. (1) and (14) using the Newmark method, as described in Sec. II.A.

The derivation of Eq. (1) with respect to the stiffness fractional change α is

$$M\frac{\partial \ddot{x}}{\partial \alpha} + C\frac{\partial \dot{x}}{\partial \alpha} + K\frac{\partial x}{\partial \alpha} = -\frac{\partial K}{\partial \alpha}x \tag{15}$$

from which the sensitivity matrix S can be obtained. Further differentiation of Eq. (15) with respect to the random variable X_p gives

$$M\frac{\partial^2 \ddot{x}}{\partial\alpha\partial X_p} + C\frac{\partial^2 \dot{x}}{\partial\alpha\partial X_p} + K\frac{\partial^2 x}{\partial\alpha\partial X_p} = -\frac{\partial M}{\partial X_p}\frac{\partial \ddot{x}}{\partial \alpha} - \frac{\partial C}{\partial X_p}\frac{\partial \dot{x}}{\partial \alpha} - \frac{\partial K}{\partial X_p}\frac{\partial x}{\partial \alpha} - \frac{\partial K}{\partial \alpha}\frac{\partial x}{\partial X_p} - \frac{\partial^2 K}{\partial\alpha\partial X_p}x \tag{16}$$

Because $\partial x/\partial X_p$ and $\partial \dot{x}/\partial X_p$ were obtained from Eq. (14); $\partial x/\partial \alpha$, $\partial \dot{x}/\partial \alpha$, and $\partial \ddot{x}/\partial \alpha$ were obtained from Eq. (15); and x was obtained from Eq. (1), $\partial^2 \ddot{x}/\partial \alpha \partial X_p$ can be finally computed from Eq. (16) to form the sensitivity matrix $\partial S/\partial X_p$:

$$\frac{\partial S}{\partial X_p} = \left[\frac{\partial^2 \ddot{x}_l}{\partial \alpha_1 \partial X_p} \quad \frac{\partial^2 \ddot{x}_l}{\partial \alpha_2 \partial X_p} \quad \cdots \quad \frac{\partial^2 \ddot{x}_l}{\partial \alpha_m \partial X_p}\right] \tag{17}$$

And $\partial \Delta\alpha/\partial X_p$ can be finally computed from Eq. (13a).

2. Uncertainty in the Exciting Force

When we consider the uncertainty in the exciting force, X_f, we also have $\partial \tilde{\ddot{x}}_d/\partial X_f = 0$, and Eq. (12) can be rewritten as

$$\frac{\partial \Delta\alpha}{\partial X_f} = -(S^T S)^{-1} S^T \left(\frac{\partial \ddot{x}_u}{\partial X_f} + \frac{\partial S}{\partial X_f} \cdot (S^T S)^{-1} S^T (\tilde{\ddot{x}}_d - \ddot{x}_u)\right) \tag{13b}$$

The derivation of Eq. (1) with respect to X_f is given as

$$M \frac{\partial \ddot{x}}{\partial X_f} + C \frac{\partial \dot{x}}{\partial X_f} + K \frac{\partial x}{\partial X_f} = D \frac{\partial \tilde{f}}{\partial X_f} - \frac{\partial K}{\partial X_f} x \tag{18}$$

We have from Eq. (9)

$$\frac{\partial \tilde{f}(t)}{\partial X_{f_i}} = [0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]^T \tag{19}$$

where only the ith component of the vector is one, with zeros in all the other components. For the initial analytical model, $\partial K/\partial X_f = 0$. But in the updated analytical model, the effect of force uncertainty has propagated in the updated model with $\alpha = \alpha^0 + \Delta\alpha^1 + \Delta\alpha^2 + \cdots$, where α is the updated vector of fractional parameter, α^0 is the initial vector of fractional parameter, and $\Delta\alpha^i$ is the vector of fractional parameter change identified after the ith iteration. Because the initial vector of fractional parameter is independent of the force uncertainty, we have $\partial \alpha^0/\partial X_f = 0$ and

$$\begin{aligned}\frac{\partial K}{\partial X_f} &= \frac{\partial K}{\partial \alpha}\frac{\partial \alpha}{\partial X_f} = \frac{\partial K}{\partial \alpha}\frac{\partial(\alpha^0 + \Delta\alpha^1 + \Delta\alpha^2 + \cdots)}{\partial X_f} \\ &= \frac{\partial K}{\partial \alpha}\left(\frac{\partial \alpha^0}{\partial X_f} + \frac{\partial(\Delta\alpha^1)}{\partial X_f} + \frac{\partial(\Delta\alpha^2)}{\partial X_f} + \cdots\right) \\ &= \frac{\partial K}{\partial \alpha}\left(\frac{\partial(\Delta\alpha^1)}{\partial X_f} + \frac{\partial(\Delta\alpha^2)}{\partial X_f} + \cdots\right)\end{aligned} \tag{20}$$

Because $\partial \Delta\alpha^i/\partial X_f \neq 0$, then $\partial K/\partial X_f \neq 0$. The sensitivity $\partial \ddot{x}_u/\partial X_f$ can be computed from Eqs. (18–20) by the Newmark method, as described in Sec. II.A.

Differentiating Eq. (15) with respect to the random variable X_f, we get

$$\begin{aligned}&M \frac{\partial^2 \ddot{x}}{\partial \alpha \partial X_f} + C \frac{\partial^2 \dot{x}}{\partial \alpha \partial X_f} + K \frac{\partial^2 x}{\partial \alpha \partial X_f} \\ &\quad = -\frac{\partial K}{\partial \alpha}\frac{\partial x}{\partial X_f} - \frac{\partial K}{\partial X_f}\frac{\partial x}{\partial \alpha} - \frac{\partial^2 K}{\partial X_f \partial \alpha} x\end{aligned} \tag{21}$$

which is similar to Eq. (12); $\partial S/\partial X_f$ can then be computed from Eqs. (18–21) using the Newmark method, and the sensitivity $\partial \Delta\alpha/\partial X_f$ can then be obtained from Eq. (13b). The effect of the uncertainty is seen propagating with iterations in the structural system through Eqs. (13a), (13b), (20), and (21).

3. Uncertainty in the Structural Response

The sensitivity of $\Delta\alpha$ with respect to the random variable $X_{\ddot{x}}$ can also be obtained similarly to that for the structural parameters. Because $\ddot{x}_u$ and S are not related with $X_{\ddot{x}}$ in the initial analytical model, $\partial \ddot{x}_u/\partial X_{\ddot{x}} = 0$ and $\partial S/\partial X_{\ddot{x}} = 0$. Equation (12) then gives

$$\frac{\partial \Delta\alpha}{\partial X_{\ddot{x}}} = (S^T S)^{-1} S^T \frac{\partial \tilde{\ddot{x}}_d}{\partial X_{\ddot{x}}} \tag{22a}$$

However, in the updated analytical model, both $\ddot{x}_u$ and S are related to $X_{\ddot{x}}$, with the uncertainties propagated in the system. Derivation of Eq. (1) with respect to $X_{\ddot{x}}$ gives

$$M \frac{\partial \ddot{x}}{\partial X_{\ddot{x}}} + C \frac{\partial \dot{x}}{\partial X_{\ddot{x}}} + K \frac{\partial x}{\partial X_{\ddot{x}}} = -\frac{\partial K}{\partial X_{\ddot{x}}} x \tag{23}$$

The sensitivities $\partial x/\partial X_{\ddot{x}}$ and $\partial \ddot{x}/\partial X_{\ddot{x}}$ can be obtained from Eq. (23) by the Newmark method, as described in Sec. II.A, and $\partial K/\partial X_{\ddot{x}}$ can be obtained in a manner similar to Eq. (20). Derivation of Eq. (14) with respect to $X_{\ddot{x}}$ gives

$$\begin{aligned}&M \frac{\partial^2 \ddot{x}}{\partial \alpha \partial X_{\ddot{x}}} + C \frac{\partial^2 \dot{x}}{\partial \alpha \partial X_{\ddot{x}}} + K \frac{\partial^2 x}{\partial \alpha \partial X_{\ddot{x}}} \\ &\quad = -\frac{\partial K}{\partial \alpha}\frac{\partial x}{\partial X_{\ddot{x}}} - \frac{\partial^2 K}{\partial \alpha \partial X_{\ddot{x}}} x - \frac{\partial K}{\partial X_{\ddot{x}}}\frac{\partial x}{\partial \alpha}\end{aligned} \tag{24}$$

and $\partial \Delta\alpha/\partial X_{\ddot{x}}$ can be written as follows, incorporating Eqs. (23) and (24) as

$$\frac{\partial \Delta\alpha}{\partial X_{\ddot{x}}} = (S^T S)^{-1} S^T \left(\frac{\partial \tilde{\ddot{x}}_d}{\partial X_{\ddot{x}}} - \frac{\partial \ddot{x}_u}{\partial X_{\ddot{x}}} - \frac{\partial S}{\partial X_{\ddot{x}}} \cdot (S^T S)^{-1} S^T (\tilde{\ddot{x}}_d - \ddot{x}_u)\right) \tag{22b}$$

Equation (20) is obtained from Eq. (10) as

$$\frac{\partial \tilde{\ddot{x}}_d(t)}{\partial X_{\ddot{x}}} = [0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]^T \tag{25}$$

and the sensitivity $\partial \Delta\alpha/\partial X_{\ddot{x}}$ is obtained from Eqs. (22) and (25). The effect of the uncertainty is seen propagating with iterations in the structural system through a modified version of Eqs. (20), (23), and (22b).

In summary, the vector of sensitivity $\partial \Delta\alpha/\partial X$ can be obtained from Eqs. (13a), (13b), and (22) as

$$\frac{\partial \Delta\alpha}{\partial X} = \left[\frac{\partial \Delta\alpha}{\partial X_p} \quad \frac{\partial \Delta\alpha}{\partial X_f} \quad \frac{\partial \Delta\alpha}{\partial X_{\ddot{x}}}\right] \tag{26}$$

D. Statistical Characteristics of the Damage Vector

The mean value of the damage vector $\Delta\alpha$ can be obtained directly from Eq. (4) as

$$E(\Delta\alpha) = (S^T S)^{-1} S^T (\ddot{x}_d - \ddot{x}_u) \tag{27}$$

because the uncertainties considered in this paper are with zero means. The damage vector $\Delta\alpha$ can be regarded as a function of the random variables, and it can be expressed as a truncated second order Taylor series as

$$\Delta\alpha(X) = \Delta\alpha(0) + \sum_{i=1}^{mt} \frac{\partial \Delta\alpha(0)}{\partial X_i} X_i + \frac{1}{2} \sum_{i=1}^{mt} \sum_{j=1}^{mt} \frac{\partial^2 \Delta\alpha(0)}{\partial X_i \partial X_j} X_i X_j \tag{28}$$

where X_i and X_j denote the ith and jth variables, respectively, and mt is the number of the random variables in the statistical analysis. The covariance matrix of $\Delta\alpha$ may be obtained as [13]

$$[\mathrm{cov}(\Delta\alpha, \Delta\alpha)]_{m \times m} \approx \left[\frac{\partial \Delta\alpha}{\partial X}\right]_{m \times mt} [\mathrm{cov}(X, X)]_{mt \times mt} \left[\frac{\partial \Delta\alpha}{\partial X}\right]^T_{m \times mt} \tag{29}$$

Because the random variables X_i and X_j $(i \neq j)$ are assumed independent, we have

$$[\mathrm{cov}(X,X)]_{mt\times mt} = \begin{bmatrix} \mathrm{cov}(X_1,X_1) & 0 & \cdots & 0 \\ 0 & \mathrm{cov}(X_2,X_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathrm{cov}(X_{mt},X_{mt}) \end{bmatrix} \quad (30)$$

It is noted that the covariance matrix of the random variables, $\mathrm{cov}(X, X)$, can be computed separately.

The random variable X_p is usually assumed to take up the following form as

$$X_p = Ep \cdot N \quad (31)$$

where Ep is a constant defining the level of variation, and N is a normally distributed vector with zero mean and unit standard deviation. We have

$$\mathrm{cov}\,(X_p, X_p) = \begin{bmatrix} (Ep)^2 & 0 & \cdots & 0 \\ 0 & (Ep)^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (Ep)^2 \end{bmatrix} \quad (32)$$

For the random variable X_f with the excitation force, it is modeled as

$$X_f = Ep \cdot N \cdot \sigma(\tilde{f}) \quad (33)$$

where $\tilde{f}$ is the vector of polluted force excitation, and $\sigma(\cdot)$ is the standard deviation of the time history. We have

$$\mathrm{cov}\,(X_f, X_f) = \begin{bmatrix} [Ep^2 \cdot \mathrm{var}(\tilde{f})] & 0 & \cdots & 0 \\ 0 & [Ep^2 \cdot \mathrm{var}(\tilde{f})] & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & [Ep^2 \cdot \mathrm{var}(\tilde{f})] \end{bmatrix} \quad (34)$$

For the random variable $X_{\ddot{x}}$, which is the measurement noise in the acceleration response, we have

$$X_{\ddot{x}} = Ep \cdot N \cdot \sigma(\tilde{\ddot{x}}_d) \quad (35)$$

Then the covariance of $X_{\ddot{x}}$ is

$$\mathrm{cov}\,(X_{\ddot{x}}, X_{\ddot{x}}) = \begin{bmatrix} [Ep^2 \cdot \mathrm{var}(\tilde{\ddot{x}}_d)] & 0 & \cdots & 0 \\ 0 & [Ep^2 \cdot \mathrm{var}(\tilde{\ddot{x}}_d)] & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & [Ep^2 \cdot \mathrm{var}(\tilde{\ddot{x}}_d)] \end{bmatrix} \quad (36)$$

Hence, Eqs. (32), (34), and (36) give the covariance matrix of all the random variables in the present study.

E. Statistical Analysis in Damage Identification

Statistical analysis on the identified results can be performed using Eqs. (27) and (29). Assuming that α^0 corresponds to the set of initial parameter changes in the analytical model, the updated set of parameter changes α^1 will be given as

$$\alpha^1 = \alpha^0 + \Delta\alpha^1 \quad (37)$$

with an expectation value of

$$E(\alpha^1) = E(\alpha^0) + E(\Delta\alpha^1) \quad (38)$$

and a covariance of

$$[\mathrm{cov}(\alpha^1, \alpha^1)]_{m\times m} = \mathrm{cov}(\alpha^0 + \Delta\alpha^1, \alpha^0 + \Delta\alpha^1) = \mathrm{cov}(\alpha^0, \alpha^0) + \mathrm{cov}(\alpha^0, \Delta\alpha^1) + \mathrm{cov}(\Delta\alpha^1, \alpha^0) + \mathrm{cov}(\Delta\alpha^1, \Delta\alpha^1) \quad (39)$$

It is noted that Eqs. (38) and (39) will remain valid during the whole process of convergence of the identified results.

F. Reliability of the Identified Damage

Once the statistical properties of the estimated stiffness changes are determined, it is possible to yield an estimate on the probability of damage. There are several techniques [13] employed to determine the probability of damage. The probability of damage quotient method is adopted for the assessment.

A normal PDF of a variable x is defined as

$$\phi(x) = \frac{1}{\sigma_x\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] \quad \text{for } -\infty < x < +\infty \quad (40)$$

where σ_x and μ_x denote the standard deviation and mean of the variable. The probability of damage is assessed by comparing the PDF of the intact structural parameter and that of the damaged parameter. Two normal PDF distributions $\phi_u(x)$ and $\phi_d(x)$ with statistical parameters (μ_u, σ_u) and (μ_d, σ_d) denote the distributions of the undamaged structural parameters and the damaged parameters, respectively, and they are shown in Fig. 1. Damage is assumed to be a loss in the local structural stiffness, with $\mu_u > \mu_d$ and $\sigma_u < \sigma_d$. To find the intersection of the two distributions, $\phi_u(x)$ and $\phi_d(x)$ are equated with combining terms to get the following quadratic equation:

$$ax^2 + bx + c = 0 \quad (41)$$

where $a = \sigma_d^2 - \sigma_u^2$, $b = 2(\mu_d\sigma_u^2 - \mu_u\sigma_d^2)$, and $c = \mu_u^2\sigma_d^2 - \mu_d^2\sigma_u^2 - 2\sigma_u^2\sigma_d^2 \ln(\sigma_d/\sigma_u)$, with roots x_1 and x_2:

$$x_1 = -\frac{1}{2a}[b + \sqrt{b^2 - 4ac}], \qquad x_2 = -\frac{1}{2a}[b - \sqrt{b^2 - 4ac}] \quad (42)$$

where x_1 is the point between μ_d and μ_u, and x_2 is between μ_u and $+\infty$, as shown in Fig. 1. The area of intersection is composed of three sections and is given by Eq. (43):

$$\begin{aligned} \mathrm{area}_1 &= \Phi\left(\frac{x_1-\mu_u}{\sigma_u}\right) - \Phi\left(\frac{-\infty-\mu_u}{\sigma_u}\right) = \Phi\left(\frac{x_1-\mu_u}{\sigma_u}\right) \\ \mathrm{area}_2 &= \Phi\left(\frac{x_2-\mu_d}{\sigma_d}\right) - \Phi\left(\frac{x_1-\mu_d}{\sigma_d}\right) \\ \mathrm{area}_3 &= \Phi\left(\frac{+\infty-\mu_u}{\sigma_u}\right) - \Phi\left(\frac{x_2-\mu_u}{\sigma_u}\right) = 1 - \Phi\left(\frac{x_2-\mu_u}{\sigma_u}\right) \end{aligned} \quad (43)$$

where $\Phi(\cdot)$ denotes the probability of a normal distribution. This area of intersection gives a probability estimate of how closely the two curves are related from a graphical viewpoint. The greater the shared area, the more related the curves become. Because the graphical area of intersection gives an indication of how related the two PDF curves are, then the graphical probability damage quotient P_D is defined as

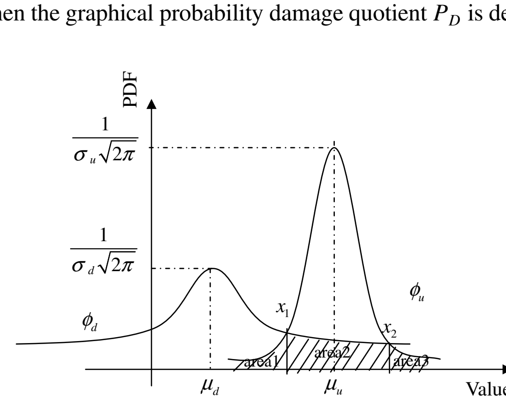


Fig. 1 Probability density functions of the structural parameter under two states.

$$P_D = 1 - (\text{area}_1 + \text{area}_2 + \text{area}_3) = \left[\Phi\left(\frac{x_1 - \mu_d}{\sigma_d}\right) + \Phi\left(\frac{x_2 - \mu_u}{\sigma_u}\right) - \Phi\left(\frac{x_1 - \mu_u}{\sigma_u}\right) - \Phi\left(\frac{x_2 - \mu_d}{\sigma_d}\right)\right] \quad (44)$$

which gives an indication of how unrelated they are. The P_D have a range of 0 to +1, where a value of 0 indicates that the two PDF curves are exactly the same (hence, no damage), and a value of 1 indicates that the stiffness values from the healthy and damaged curves are not related in any way, indicating that damage has occurred. Every element of the structure should be analyzed in this manner to give a P_D in accordance with Eq. (44).

III. Numerical Study

A. Structure

A three-dimensional, five-bay, steel-frame structure is shown in Fig. 2. The finite element model consists of 37 three-dimensional Euler beam elements and 17 nodes. The length of all the horizontal, vertical, and diagonal tube members between the centers of two adjacent nodes is exactly 0.5 m. The structure orientates horizontally and is fixed into a rigid concrete support at three nodes at one end. Table 1 gives a summary of the main material and geometrical properties of members of the frame structure. Each node has six DOF, and altogether there are 102 DOF for the whole structure.

The translational and rotational restraints at the supports are represented by large stiffnesses of 1.5×10^{11} kN/m and 1.5×10^{10} kN · m/rad, respectively, in the translation and the rotational DOF. Rayleigh damping is adopted for the system with $C = \alpha_1 M + \alpha_2 K$, where α_1 and α_2 are constants to be determined from two given damping ratios corresponding to two unequal frequencies of vibrations f_1 and f_2. The two constants are given as

$$\alpha_1 = 2f_1 f_2(f_2\xi_1 - f_1\xi_2)/(f_2^2 - f_1^2)$$

and

$$\alpha_2 = 2(f_2\xi_2 - f_1\xi_1)/(f_2^2 - f_1^2)$$

The two damping ratios for the first two modal frequencies are taken as $\xi_1 = 0.01$ and $\xi_2 = 0.005$. The first 12 natural frequencies of the structure are 9.21, 28.26, 33.71, 49.01, 49.72, 71.02, 89.80, 153.93, 194.33, 209.80, 256.51, and 274.82 Hz from the eigenvalue analysis of the structure. The sampling frequency is 2000 Hz.

B. Damage Identification

A sinusoidal excitation is applied onto the structure at the eighth node in the z direction with an amplitude of 3 N and at a frequency of 30 Hz. The acceleration response computed at the fifth node in the z direction is taken as the measured response, and the first 500 data points are used to identify the damage. A damage case with 5 and 10% reduction of the flexural stiffness in the 7th and 26th members, respectively, is studied. The effect of each type of uncertainty on the damage identification will be discussed next.

1. *Uncertainty with the Mass Density*

The uncertainty of mass density of material is assumed to have 1% amplitude with $Ep = 0.01$ in Eq. (31), and damage identification is performed on the structure using the proposed approach. The mean

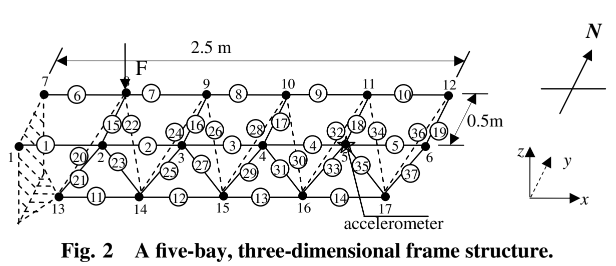


Fig. 2 A five-bay, three-dimensional frame structure.

Table 1 Material and geometric properties

Properties	Value
Young's modulus, N/m^2	2.10E11
Area, m^2	6.597E-5
Density, kg/m^3	1.2126E4
Mass, kg	0.32
Poisson ratio	0.3
Moment of area I_y, m^4	3.645E-9
Moment of area I_z, m^4	3.645E-9
Torsional rigidity J, m^4	7.290E-9

stiffness changes $\mu_{\Delta\alpha}$ obtained from Eq. (4) for all the elements in the structure are shown in Fig. 3a after the first and second iterations of computation. The standard deviations of the stiffness change, $\sigma_{\Delta\alpha}$, for all the elements obtained from the proposed method after the first and second iterations are shown in Fig. 4. They are compared with those computed from the Monte Carlo method [19], from 1000 samples of data. This is done by adding random noise to the mass density, according to the assumed distribution. Deterministic damage detection is then performed to obtain the damage vector. The procedure is performed 1000 times to have 1000 sets of identified results from which the statistical characteristics can be computed. The two sets of standard deviations are very close, indicating that the proposed statistical method is correct. The standard deviations from the first iteration range from 0.96 to 3.34% of the identified flexural stiffness of member when there is only 1% amplitude in the variation of the mass density of the analytical model. This indicates that the uncertainty amplifies the error in the identified results. The standard deviations in Fig. 4b after the second iteration range from 1.01 to 3.77%, and they are of similar magnitude to those after the first iteration. This indicates that the amplifying effect of the variation in the mass density is not significant in the second iteration, because the bulk of the damage vector was updated in the first iteration, as shown in Fig. 3a.

To study the effect of the variation throughout the iterations, 10 iterations are performed, and the mean values of the damaged members are shown in Figs. 3b and 3c. The standard deviations for the 7th, 18th, and 26th members are shown in Fig. 5. The damage parameters converge quickly to the true values with only four iterations. The standard deviations in Fig. 5 also converge quickly to a constant with increasing iterations in all the elements. Because the uncertainty with the mass density of the finite element model cannot be represented in the updated results, its detrimental effect on the model updating is carried forward to the next iteration in the form of variation in the identified result. It is only after four iterations that the correct reductions in the 7th and 26th members were updated, and the variation of the identified results becomes stabilized with a constant standard deviation.

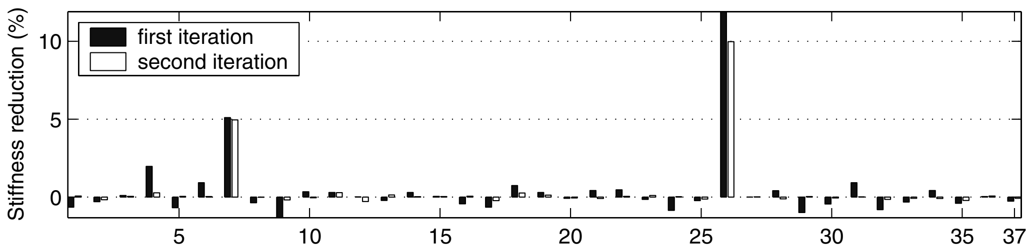
Equation (29) shows that the standard deviation of the damage vector is linearly related to the amplitude of the uncertainty Ep. Only the computation for the case with a unit variation is necessary, because it can be easily extended to other cases with different variation amplitudes. It should be noted that all the statistical analysis in this study is restricted to a small variation, such that the linear approximation assumption in Eq. (2) is valid.

2. *Uncertainty with the Elastic Modulus of Material*

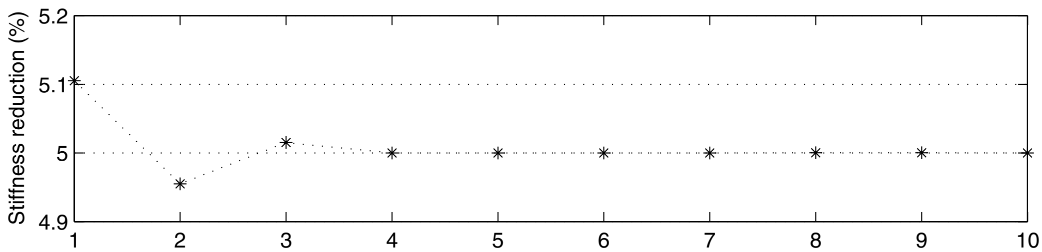
When 1% variation is included in the elastic modulus of material of the initial analytical model, the mean values of the identified flexural stiffness are the same as those shown in Fig. 3, because they are also computed from Eq. (4). The standard deviation of the identified results is shown in Fig. 6, along with those obtained from the Monte Carlo technique. The standard deviation for all members has a maximum value of 1.68% after the first iteration, and it drops to 0.26% after the second iteration. This shows the following:

1) The effect of random variation in the elastic modulus is comparable with that of the mass density after the first iteration

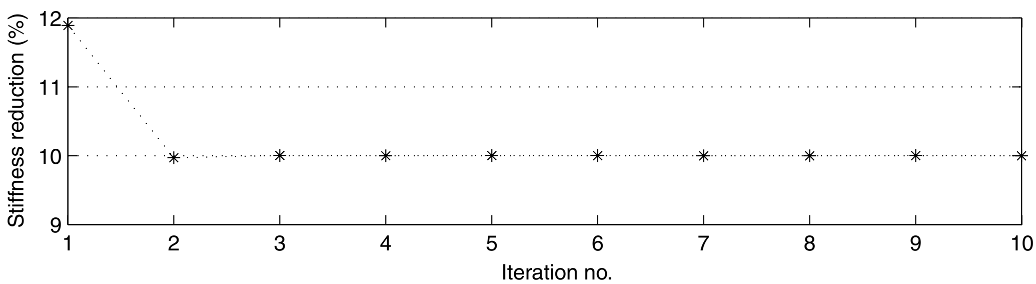
2) The significant reduction after the second iteration shows that the flexural stiffness, which is closely related to the elastic modulus,



a) Mean values after first iteration and second iteration

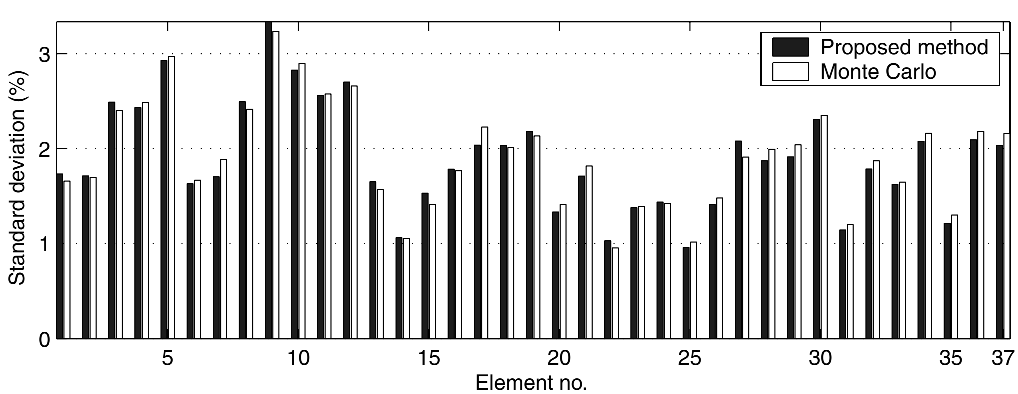


b) Evolution of mean value for the 7th element

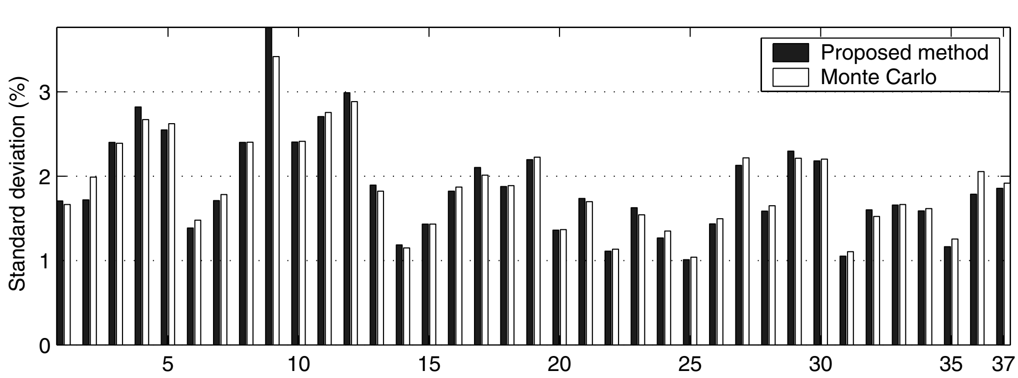


c) Evolution of mean value for the 26th element

Fig. 3 Mean value of identified results and the evolution of mean value with iterations for the 7th and 26th elements.



a) First iteration



b) Second iteration

Fig. 4 Standard deviation of identified results due to uncertainty in the mass density.

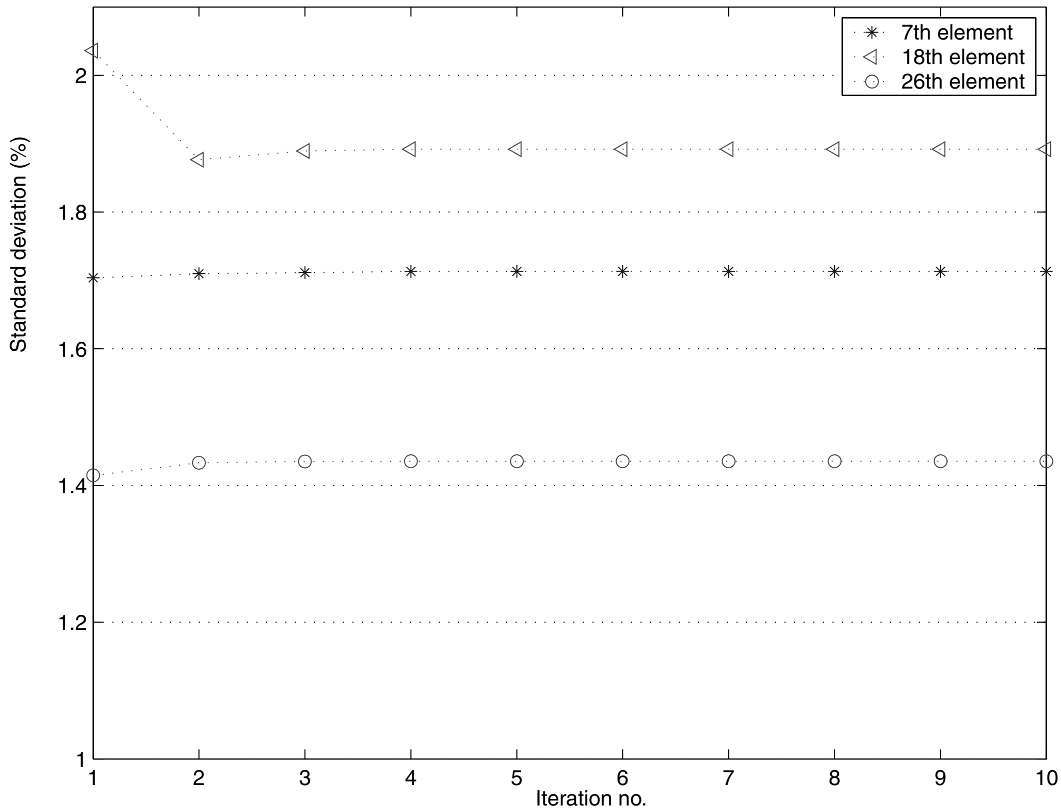
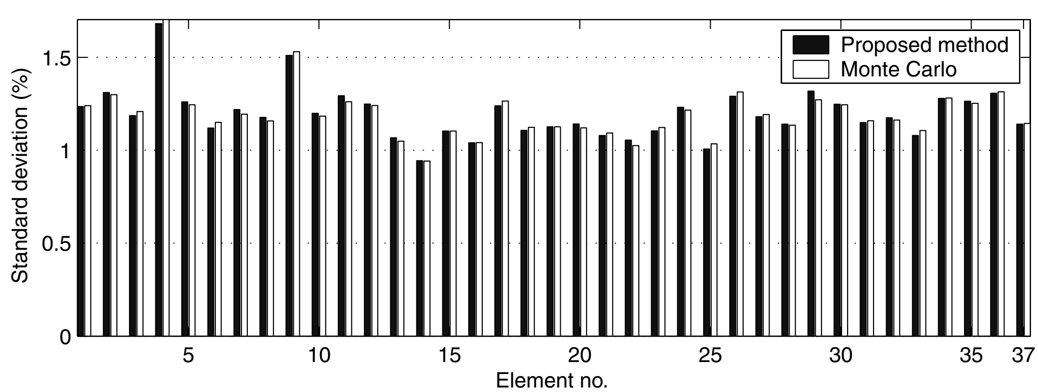
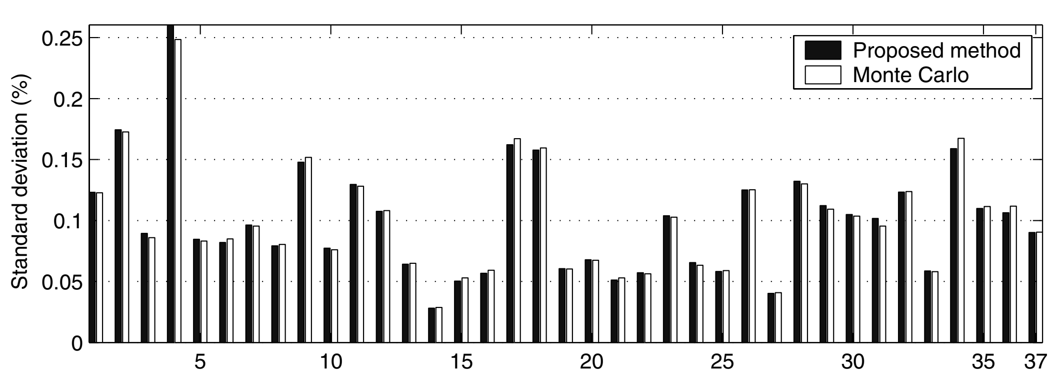


Fig. 5 Evolution of standard deviation with iterations due to uncertainty in the mass density.



a) First iteration



b) Second iteration

Fig. 6 Standard deviation of identified results due to uncertainty in the elastic modulus of material.

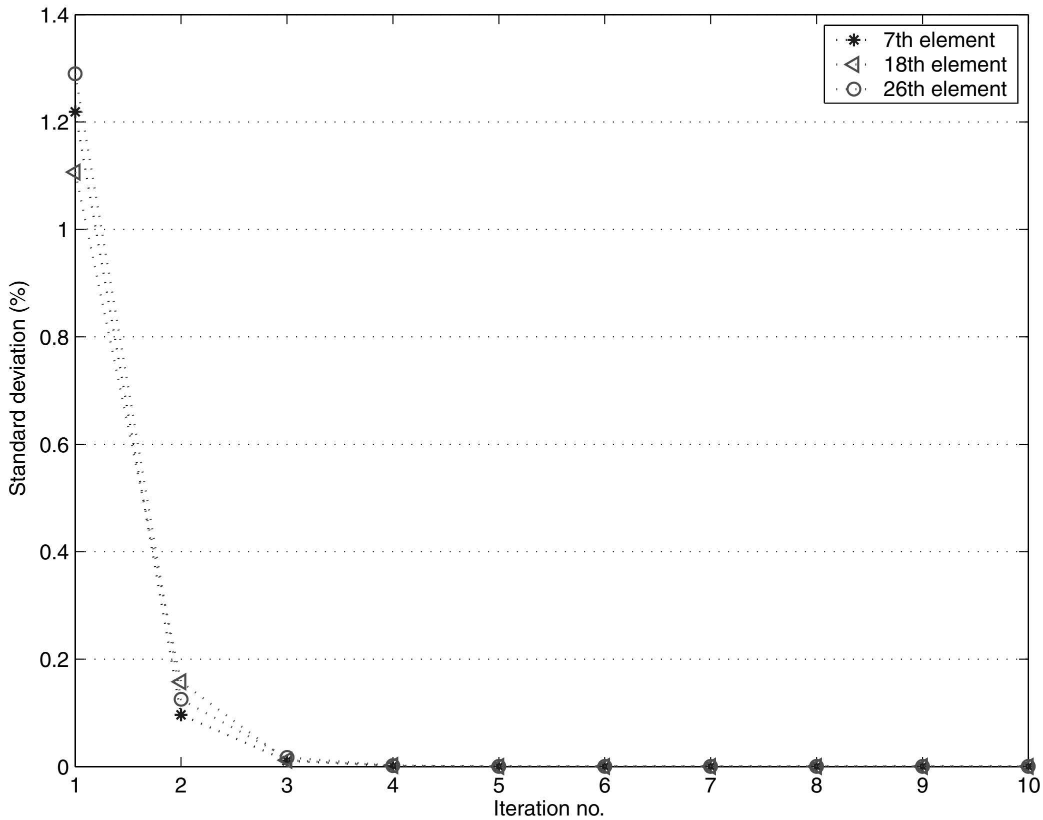
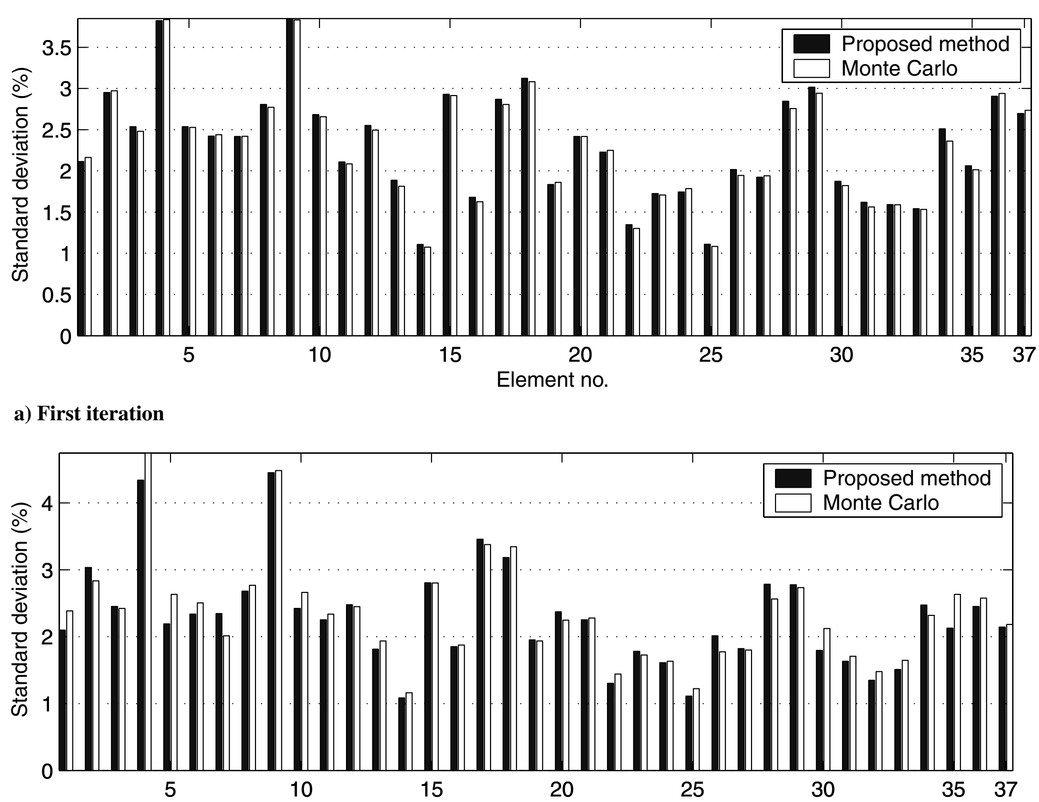


Fig. 7 Evolution of standard deviation with iterations due to uncertainty in the elastic modulus of material.



a) First iteration

b) Second iteration

Fig. 8 Standard deviation of identified results due to uncertainty in the force excitation.

was updated with the mitigation of the associated variation effect in the subsequent iterations.

To study the effect of variation with iterations, 10 iterations are performed; the mean values of the identified results are shown in Figs. 3b and 3c, and the standard deviations for the 7th, 18th, and 26th members are shown in Fig. 7. The flexural stiffness converges quickly to the true values with only four iterations, and the standard deviations in Fig. 7 also reduce quickly to zero with about four

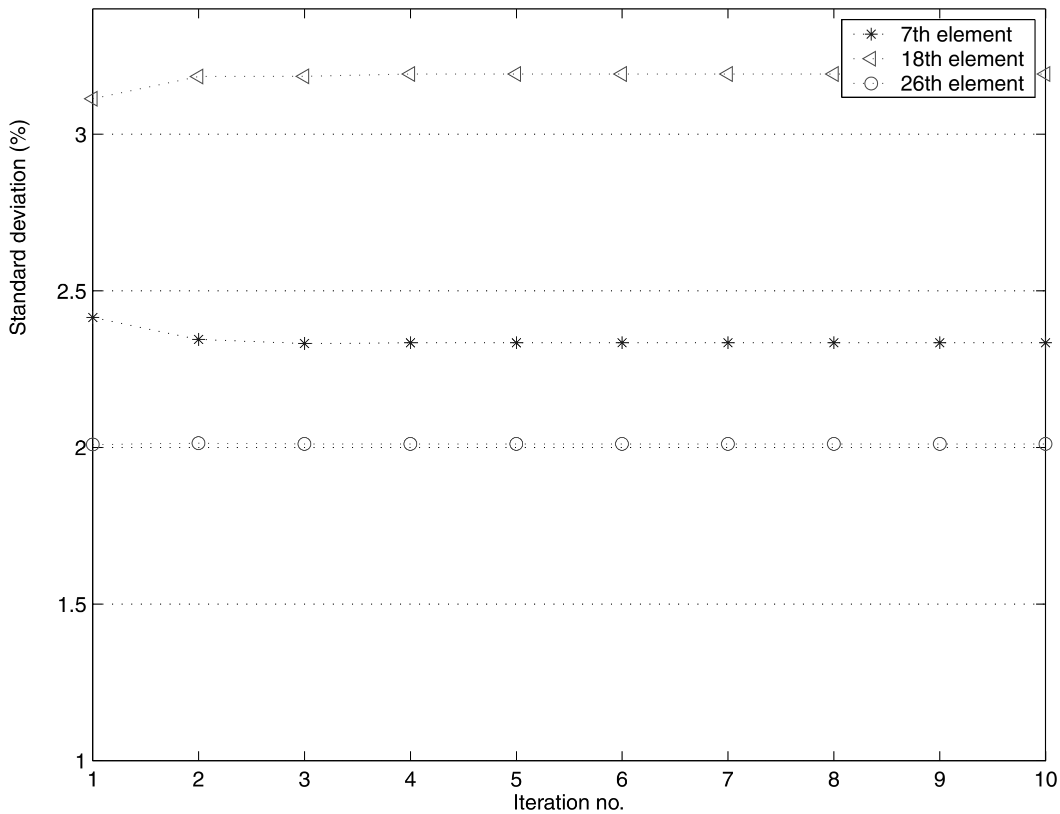


Fig. 9 Evolution of standard deviation with iterations due to uncertainty in the force excitation.

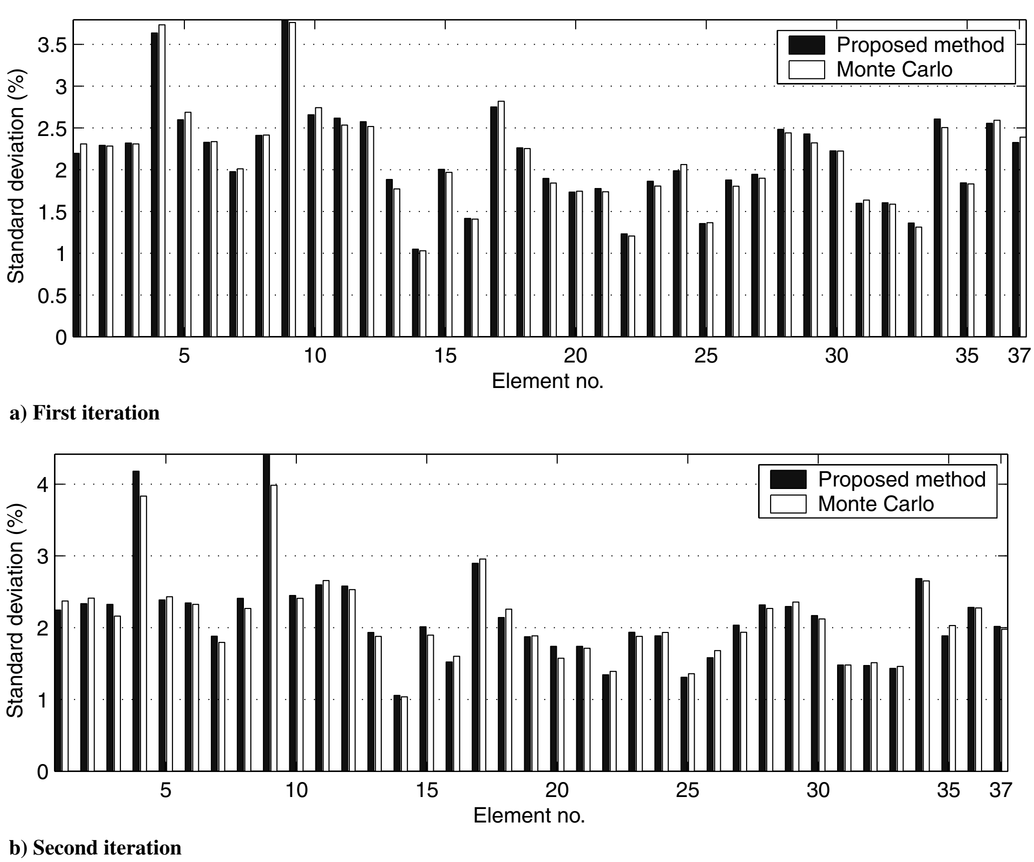


Fig. 10 Standard deviation of identified results due to uncertainty in the measured response.

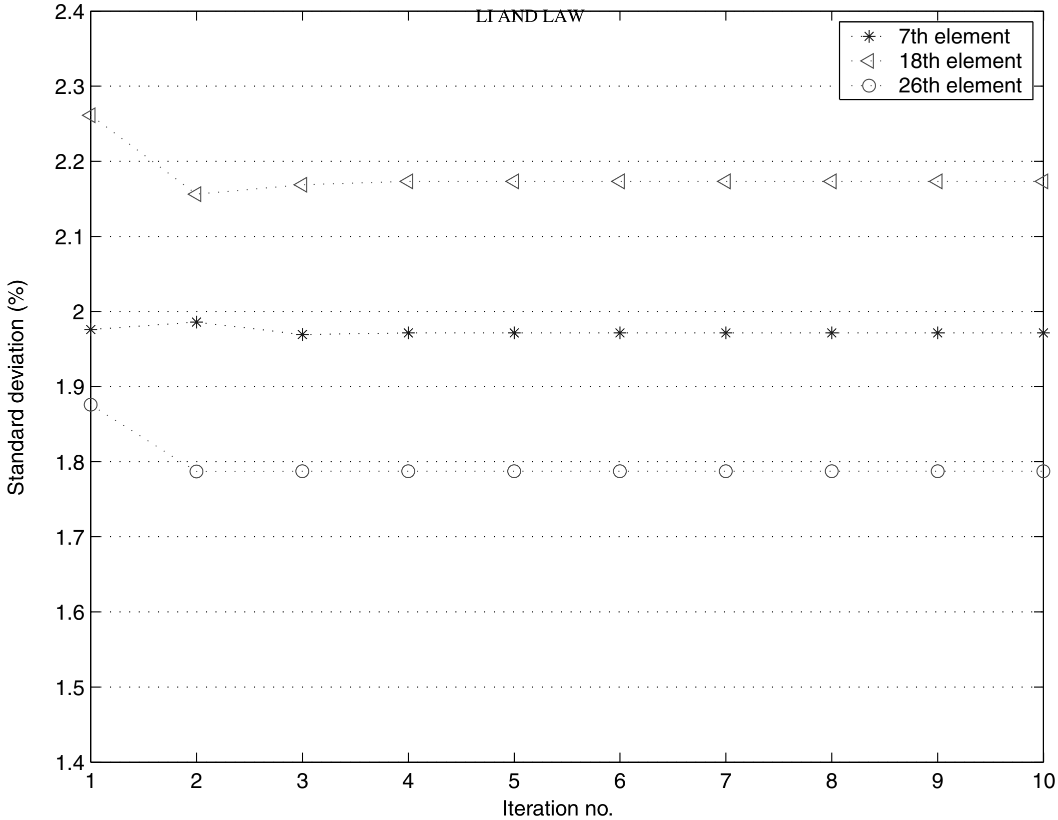


Fig. 11 Evolution of standard deviation with iterations due to uncertainty in the measured response.

iterations. The standard deviation for other undamaged members also exhibits this behavior. This indicates that the effect of variation in the stiffness parameter can be fully represented in the updated model after a few iterations, with no residual uncertainties in the updated finite element model.

3. Uncertainty with the Excitation Force

One percent variation, as defined in Eq. (33), is included in the excitation force for the dynamic analysis in the intact state. The mean values of the identified flexural stiffness are the same as those in Fig. 3, and the standard deviations of the identified results after the first and second iterations are shown in Fig. 8, along with those from the Monte Carlo technique. The standard deviation of the identified results after the first iteration has a maximum value of 3.85%, which increases slightly to 4.45% after the second iteration. The error in the damage vector was amplified by the 1% variation in the force excitation. To further investigate the propagation of this uncertainty in the identified results throughout the updating process, 10 iterations are performed, and the standard deviations for the 7th, 18th, and 26th members are shown in Fig. 9. The standard deviations for the three members converge quickly to a constant in four iterations, similar to the uncertainty in the mass density. This observation indicates the following:

1) The effect of variation with the excitation force is similar to that with the mass density, but larger. The flexural stiffness of the damaged member can be updated to have a stable standard deviation in the identified results with iteration.

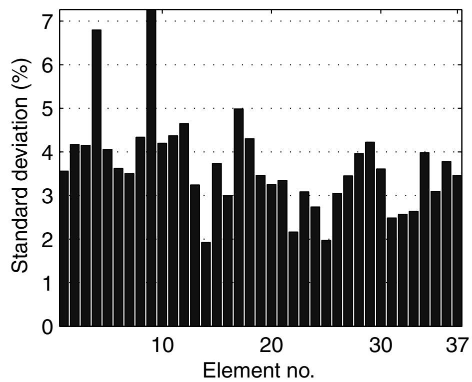
2) The bulk of the damage vector was updated with the variation effect accounted for in the first few iterations, as shown in Fig. 9.

4. Uncertainty with the Measured Structural Response

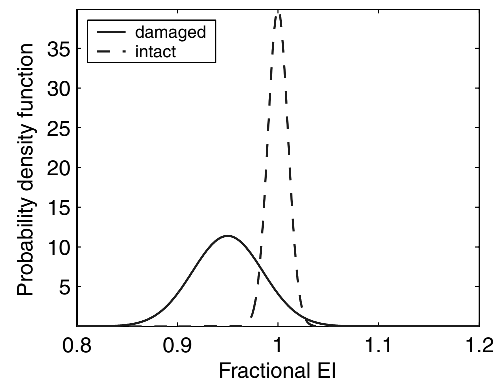
A 1% random noise, as defined in Eq. (35), is added into the measured acceleration responses from the damaged structures to simulate the measurement noise. The standard deviation of all the identified results after the first iteration is shown in Fig. 10a with a maximum value of 3.79%, which increases to 4.42% after the second iteration (shown in Fig. 10b). The error in the identified results was amplified by the 1% noise in the measured acceleration responses. The standard deviation for the three selected members in all 10 iterations shown in Fig. 11 indicates that the propagation of the

Table 2 Identified damage probability and mean value of damage index

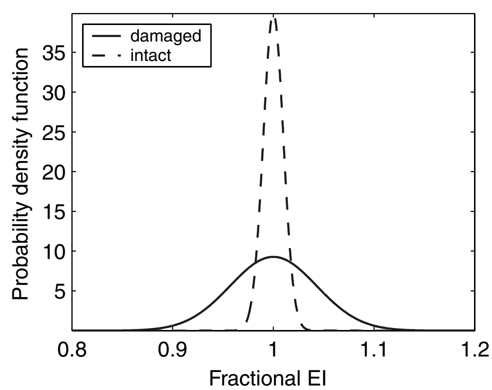
Member no.	Mean value of α_i, %	Damage probability, %
1	0.0	54.4
2	0.0	59.5
3	0.0	59.3
4	0.0	72.3
5	0.0	58.6
6	0.0	55.0
7	5.0	79.4
8	0.0	60.6
9	0.0	73.8
10	0.0	59.7
11	0.0	60.9
12	0.0	62.7
13	0.0	51.2
14	0.0	30.5
15	0.0	56.0
16	0.0	48.4
17	0.0	64.6
18	0.0	60.4
19	0.0	53.5
20	0.0	51.3
21	0.0	52.3
22	0.0	35.6
23	0.0	49.4
24	0.0	45.0
25	0.0	31.6
26	10.0	98.8
27	0.0	53.4
28	0.0	57.9
29	0.0	59.8
30	0.0	54.8
31	0.0	41.2
32	0.0	42.6
33	0.0	43.6
34	0.0	58.0
35	0.0	49.5
36	0.0	56.4
37	0.0	53.4



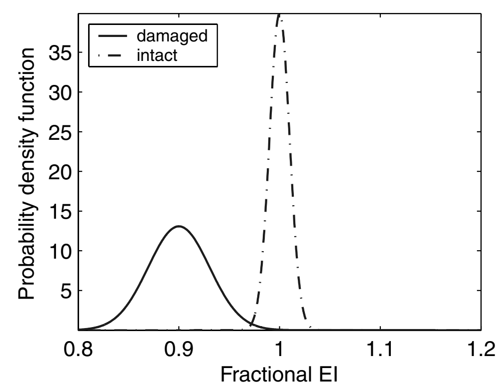
a) Standard deviation



b) PDF for 7th element



c) PDF for 18th element



d) PDF for 26th element

Fig. 12 Identified results with all types of uncertainties.

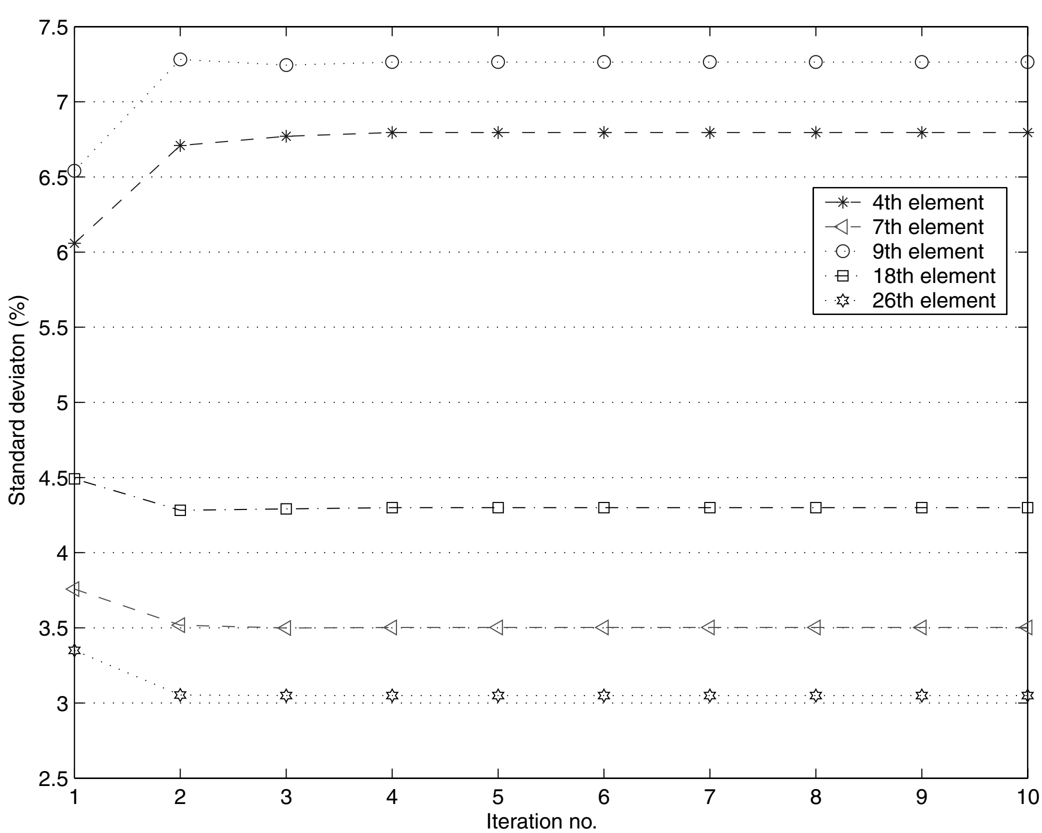


Fig. 13 Evolution of standard deviations with iterations due to all uncertainties under study.

standard deviation is similar to that observed with variation in the mass density. Other discussions are similar to those for the effect of variation in the mass density and the excitation force.

As a summary to the preceding discussions, the effect of variation in the flexural stiffness of the member is the least significant, and it can be mitigated with increasing iterations of model updating. The effect of variation in the mass density of the finite element is more significant, and those for the excitation force and the structural response are highly significant.

C. Reliability Assessment Including All Uncertainties

When all the different types of uncertainties (variations in the mass density, elastic modulus, excitation force, and measured acceleration response) with an amplitude of 1% are included, the same damage case as for the previous investigations is studied again for a reliability assessment. The same damage-identification approach is used, and the updating process needs only five iterations to arrive at a set of stable results. The mean values after five iterations are shown in Table 2, and the standard deviation of the identified results and the PDF curves for the 7th, 18th, and 26th members are shown in Fig. 12. The two largest standard deviations are 6.80 and 7.26% at the 4th and 9th members, respectively, indicating a large effect from the variations of the different parameters included in the study.

The PDF curves of the three members clearly show that the distribution becomes flatter when variations are included and the intersection area of the PDF curves from the intact and damaged states becomes smaller, leading to an increase in the probability of damage or a false positive in the undamaged member. The identified probability of damage for all the members is also shown in Table 2, in which the 26th member has the largest value of 98.8%, indicating that the member is highly likely to be damaged. The second-largest value of 79.4% occurs at the seventh member, indicating this member is very likely to be damaged. The 4th and 9th members also have a high possibility of damage of 73.8 and 72.3%, respectively, but they can be categorized as false positives in the damage identification, because they also correspond to the two largest standard deviations of the identified results. From Fig. 13, the standard deviations of elements 4, 7, 9, 18, and 26 vary in the first several iterations and become stable afterward. Elements 4 and 9 are symmetrically located on the structure. The standard deviations of elements 4 and 9 are relatively larger than those of other elements, probably because they are located in the segment length of the structure close to the measurement location; however, the exact reason deserves further study. The proposed method can be concluded to be able to identify local damage successfully with a few false positives in the structure when uncertainties in the analytical model and the measured dynamic responses are included in the analysis.

It should be noted that the assumptions of zero mean and normal distribution for the uncertainties are the basis of the present study. If the zero mean assumption is not valid, Eq. (27) is also not valid. If the normal random distribution is not satisfied, the distribution of the identified parameter will not be normal distribution and this will add difficulties to the reliability analysis with the identified results. The computation cost of the proposed method is as high as 6–7 h when all the uncertainties studied are included to compute the damage vector in 10 iterations with a Pentium 4 PC working on a CPU with 2.66 GHz and 1 G of RAM. The computation cost of the Monte Carlo method is still higher (approximately two times longer) for the same computation.

IV. Conclusions

This paper proposes a statistical method for damage identification in structures with uncertainties in the analytical model, the excitation, and the measured dynamic characteristics. The analytical formulation of these variations of the different system parameters, the excitation, and the measured responses with respect to the identified results are given. The uncertainties from the mass density, elastic modulus, and excitation force are required in the computation with the analytical model, whereas that of the measured acceleration responses from the damaged structure is studied separately. Results show that the uncertainties in the system parameters remain in the updating process in the form of a standard deviation in the identified results. The standard deviation does not diverge when the bulk of the damage vector is obtained after a few iterations. In the simulation study with a three-dimensional, five-bay, steel-frame structure with uncertainties, reliability assessment is performed based on the statistical analysis, and the identified results show that the proposed method can identify damage successfully, with a few false positives in the structure with a probability prediction.

Appendix: The Newmark Method

The basic principles in the Newmark method can be described as

$$\dot{x}_{n+1} = \dot{x}_n + (1-\gamma)\ddot{x}_n \cdot \Delta t + \gamma \ddot{x}_{n+1} \cdot \Delta t \tag{A1}$$

$$x_{n+1} = x_n + \dot{x}_n \cdot \Delta t + \frac{1-2\beta}{2}\ddot{x}_n \cdot \Delta t^2 + \beta \ddot{x}_{n+1} \Delta t^2 \tag{A2}$$

where Δt is the time step used in the computation, subscripts n and $n+1$ denote the time instances, and γ and β are constants. The displacement, velocity, and acceleration time histories of Eq. (1) can be computed from the preceding equations. Zero initial values are assumed in this study. A stable solution can be obtained when $\gamma \geq 0.5$ and $\beta \geq 0.25(0.5+\gamma)^2$. If $\gamma = 0.5$ and $\beta = 0.25$, the method becomes the well-known averaging acceleration method for the solution of the differential equation of motion. There may be a problem with the stability of this method with noise effect, but this is not covered in this paper.

Acknowledgment

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